Tutorial 10

Let $\mathcal{A} = \{1, \cdots, n\}$ be the set of players and ν be a characteristic form.

Null player

Player i is said to be a null player of ν if

$$\nu(S \cup \{i\}) = \nu(S), \text{ for any } S \subseteq \mathcal{A} \setminus \{i\}.$$

Symmetric players

Two players i and j are said to be symmetric if

$$\nu(S \cup \{i\}) = \nu(S \cup \{j\}), \text{ for any } S \subseteq \mathcal{A} \setminus \{i, j\}.$$

Simple games

A nonzero cooperative game $(\mathcal{A}, \nu)$ is said to simple if

$$\nu(S) = 0 \text{ or } 1, \text{ for any } S \subseteq \mathcal{A}.$$

Exercise 1. *Let $(\mathcal{A}, \nu)$ be a simple game with $\mathcal{A} = \{1, \cdots, n\}$. A player i is said to be a veto player if $\nu(\mathcal{A} \setminus \{i\}) = 0$. Prove that $C(\nu) \neq \emptyset$ if and only if there is at least one veto player.*

Soluton. "$\Leftarrow$". Assume that there is a veto player. Without lose of generality, let us say Player 1 is a veto player. Then $\nu(\mathcal{A} \setminus \{1\}) = \nu(\{2, \cdots, n\}) = 0$. By the superadditivity of ν, we have $\nu(S) = 0$ for any $S \subseteq \{2, \cdots, n\}$. Take $x_1 = 1, x_2 = \cdots = x_n = 0$. It is direct to check that $(1, 0, \cdots, 0) \in C(\nu)$. In particular, $C(\nu) \neq \emptyset$.

"$\Rightarrow$". Assume that there are no veto players. Then,

$$\nu(\mathcal{A} \setminus \{i\}) = 1, \text{ for } i = 1, \cdots, n.$$

Assume $(x_1, \cdots, x_n) \in C(\nu)$. Then, by the characterization of the core,

$$1 = \sum_{i=1}^{n} x_i = x_i + \sum_{j \neq i} x_j \geq x_i + \nu(\mathcal{A} \setminus \{i\}) = x_i + 1 \geq 1,$$

which implies $x_i = 0$ for $i = 1, \cdots, n$. This contradicts with the fact that $\sum_i^n x_i = 1$. Hence $C(\nu) = \emptyset$.

Exercise 2. *Let $(\mathcal{A}, \nu)$ be a cooperative game, where $\mathcal{A} = \{1, \cdots, n\}$. Let*

$$\delta_i = \nu(\mathcal{A}) - \nu(\mathcal{A} \setminus \{i\}), i = 1, \cdots, n.$$

Prove that if $\sum_{i=1}^n \delta_i < \nu(\mathcal{A})$, then the core $C(\nu) = \emptyset$.

Solution. Assume that $(x_1, \cdots, x_n) \in C(\nu)$. Then,

$$x_1 + \cdots + x_n = \nu(\mathcal{A}). \tag{1}$$

By the characterization of the core, we have

$$\sum_{j \neq i} x_j \geq \nu(\mathcal{A} \setminus \{i\}) = \nu(\mathcal{A}) - \delta_i, i = 1, \cdots, n.$$

Then take summation with respect to i and use (1), we have

$$(n-1)\nu(\mathcal{A}) \geq n\nu(\mathcal{A}) - \sum_{i=1}^{n} \delta_i > (n-1)\nu(\mathcal{A}).$$

A contradiction. Hence $C(\nu) = \emptyset$.

Exercise 3. *Let $\mathcal{A} = \{1, 2, 3\}$ be the set of the players and ν be a characteristic function. Assume that $\nu(\{1\}) = a$, $\nu(\{2\}) = \nu(\{3\}) = \nu(\{2, 3\}) = 0$, $\nu(\{1, 2\}) = b$, $\nu(\{1, 3\}) = \nu(\{1, 2, 3\}) = c$ $(a \leq b \leq c)$. Find the core $C(\nu)$ in terms of a, b, c and show that it is contained in a line segment in $\mathbb{R}^3$.*

Solution. By the characterization of the core $C(\nu)$, a point $(x_1, x_2, x_3) \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq a, x_2 \geq 0, x_3 \geq 0, \\ x_1 + x_2 \geq b, x_1 + x_3 \geq c, x_2 + x_3 \geq 0, \\ x_1 + x_2 + x_3 = c. \end{cases}$$

These inequalities are equivalent to

$$\begin{cases} x_1 \geq \max\{a, b\} = b, x_2 = 0, x_3 \geq 0, \\ x_1 + x_3 = c. \end{cases}$$

Hence the core is

$$C(\nu) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq b, x_2 = 0, x_3 \geq 0, x_1 + x_3 = c\}.$$

It is clear that $C(\nu)$ is contained in the intersection of two planes $x_2 = 0$ and $x_1 + x_2 + x_3 = c$ in $\mathbb{R}^3$. Hence $C(\nu)$ is in a line segment in $\mathbb{R}^3$.